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Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces

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Abstract

We present an optimal $\Theta(n)$-time algorithm for the selection of a subset of the vertices of an n-vertex plane graph G so that each of the faces of G is covered by (i.e., incident with) one or more of the selected vertices. At most $\lfloor n/2 \rfloor$ vertices are selected, matching the worst-case requirement. Analogous results for edge-covers are developed for two different notions of "coverage". In particular, our linear-time algorithm selects at most $n-2$ edges to *strongly* cover G, at most $\lfloor n/3 \rfloor$ diagonals to cover G, and in the case where G has no quadrilateral faces, at most $\lfloor n/3 \rfloor$ edges to cover G. All these bounds are optimal in the worst-case. Most of our results flow from the study of a relaxation of the familiar notion of a 2-coloring of a plane graph which we call a face-respecting 2-coloring that permits monochromatic edges as long as there are no monochromatic faces. Our algorithms apply directly to the location of guards, utilities or illumination sources on the vertices or edges of polyhedral terrains, polyhedral surfaces, or planar subdivisions.

1. Introduction

In 1973, Victor Klee posed the problem of determining the minimum number of guards or light sources sufficient to cover or illuminate the interior of an n-sided art gallery modeled by a simple polygon. Using a combinatorial argument [5], Chvátal showed that $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary. Subsequently, Fisk [9] gave a concise and elegant proof of the same result, using the fact that the vertices

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of a triangulated polygon can be three-colored. Since then, there has been an explosion of results in this area, where many different variations of this original problem have been studied. The reader is referred to the book by O'Rourke [12], the survey by Shermer [16], and the survey by Urrutia [17] for an overview of the area.

Most of the research to date has been concerned with guarding the boundary of objects in two dimensions. Comparatively little is known about guarding only portions of objects or guarding objects in three dimensions. A modest step in the latter direction is the study of guarding polyhedral terrains. This problem was first investigated by deFloriani et al. [7] who showed that the minimum number of point guards can be found using a set covering algorithm. Cole and Sharir [6] showed that determining the minimum number of guards for a polyhedral terrain is NP-hard. Goodchild and Lee [10] and Lee [11] presented some heuristics for placing guards at a subset of the vertices of a polyhedral terrain.

In the same spirit as the original work on guarding of art galleries, recent efforts have been made to bound the number of guards required in the worst case as a function of the number of vertices of the polyhedral terrain. Bose et al. [3] showed that $\lfloor n/2 \rfloor$ vertex-guards are always sufficient and sometimes necessary to guard the entire surface of an n-vertex triangulated polyhedral terrain. With respect to edge-guards, i.e., guards free to patrol an entire edge of the terrain, they establish that at least $\lfloor (4n-4)/13 \rfloor$ edge-guards are necessary for the same task, in the worst case. The complementary result that $\lfloor n/3 \rfloor$ edge-guards are always sufficient was proved by Everett and Rivera-Campo [8]. Both sufficiency results are based on the existence of four-colorings of planar graphs, for which practical algorithms are not known to exist. In [3], $O(n)$ time algorithms based on efficient five-coloring algorithms are presented for placing $\lfloor 3n/5 \rfloor$ vertex-guards or $\lfloor 2n/5 \rfloor$ edge-guards to cover an n-vertex triangulated polyhedral terrain.

Works to date on guarding triangulated polyhedral terrains typically suppress the geometric aspects of the problem, focusing instead on the underlying combinatorial problem of covering the faces of the associated maximal plane graph. A vertex v of a plane graph G is said to *cover* all of the faces of G incident on v. An edge *covers* the union of the faces covered by its associated vertices. The more familiar combinatorial relationship between an edge and its (at most two) incident faces is referred to as *strong* coverage. The correspondence between the geometric and combinatorial aspects of the problem is based on the observation that, in any polyhedral terrain whose faces are all convex, the visible region associated with a guard contains the union of all faces adjacent to the points visited by that guard. So in the case of a vertex guard, it is the faces of the graph adjacent to that vertex and for an edge-guard, it is the faces of the graph adjacent to one of the two endpoints of the edge. Furthermore, when the polyhedral terrain is convex, the visible region contains nothing more. Therefore, upper bounds on the number of guards or light sources needed for a triangulated polyhedral terrain are provided by upper bounds on the number of vertices or edges (viewed as isolated points or point pairs respectively) needed to cover the corresponding subset of triangles of the associated plane graph.

For general, i.e., not necessarily triangulated polyhedral terrains with convex faces, it is reasonable to generalize the notion of an edge-guard to that of a face-diagonal-guard, hereafter simply diagonal-guard. Diagonal guards are modeled by point pairs constrained to share a common face of the underlying plane graph. Other utility location applications motivate a somewhat stronger notion of "coverage" reflecting the static nature of utility placements. As an illustration of such applications consider the following. Suppose that one is given the floor plan of an office complex modeled as a planar subdivision where the faces are rooms and the edges are walls and corridors, and one wishes to place electrical connections in a specified subset of the rooms, but minimize the number of walls containing such connections. This can be done by strategically placing them in walls and corners shared by many rooms. Hence, a minimum set of edges suitably covering all the specified faces of the planar subdivision represents a placement of electrical connections that is accessible to every room while in some sense minimizing the number of sources needed to provide such connections.

In this paper, we focus on the problem of finding worst-case-optimal covers of plane graphs. Given an n-vertex (not necessarily triangulated) plane graph G we present $\mathrm{O}(n)$-time algorithms that:

(i) cover G using at most $\lfloor n/2 \rfloor$ vertices;
(ii) cover G using at most $\lfloor n/3 \rfloor$ diagonal-guards;
(iii) cover G using at most $\lfloor n/3 \rfloor$ edges, provided G contains no quadrilateral faces; and
(iv) strongly cover G using at most $n - 2$ edges.

In each case, there exist n-vertex plane graphs whose minimum cover has the specified size.

We describe the implications of these results for the location of utilities, guards and light sources in polyhedral terrains, polyhedral surfaces and planar subdivisions. Most of our results flow from the study of a relaxation of the familiar notion of a 2-coloring of a plane graph which we call a face-respecting 2-coloring that permits monochromatic edges as long as there are no monochromatic faces. Most of the geometric and graph theoretic terminology in what follows is standard; for additional details, we refer the reader to [13] and [4].

2. Face-respecting 2-colorings

A graph G is *planar* if it has a *planar embedding*, that is it can be drawn in the plane so that its edges intersect only at their endpoints. An embedded planar graph is referred to as a *plane* graph. A plane graph each of whose edges is embedded as a straight line segment is called a *planar subdivision*.

A *k-vertex-coloring* of a graph G is an assignment of one of k distinct colors to each vertex of G. Such a coloring is *valid* if no two adjacent vertices have the same color. Here is where we break from tradition and call such a coloring an *edge-respecting* coloring since no edge is monochromatic, i.e., colored the same at both endpoints. This allows us to define a natural relaxation of the notion of a valid 2-coloring for plane graphs. Specifically, if G is a plane graph then we say that a k-coloring is *face-respecting* if no face of G is monochromatic, i.e., has all of its vertices colored the same. Face-respecting 2-colorings of plane graphs play a fundamental role in our covering algorithms for plane graphs. This section develops our results for face-respecting 2-colorings. Their application to vertex-, edge- and diagonal-covers of plane graphs is taken up in the next section.

A plane graph G' is a *refinement* of the plane graph G if the edges of G form a subset of the edges of G'. A plane graph is *maximal* (also called a *triangulation*) if it has no non-trivial refinement (i.e., all of its faces are triangles). As a consequence of the following, it will suffice to show how to construct face-respecting 2-colorings of maximal plane graphs.

Observation 2.1. *A face-respecting 2-coloring of any refinement of a plane graph G is also a face-respecting 2-coloring of G.*

The four-color theorem [1] states that every planar graph has an edge-respecting 4-coloring. Combining the colors of an edge-respecting 4-coloring into two color pairs, it is easy to see that no triangle is colored by vertices from one pair alone, and hence every maximal plane graph admits a face-respecting 2-coloring [3].

This implication is not computationally satisfying since simple constructive proofs of the four-color theorem have not been demonstrated. Haken and Appel [1] presented an $O(n^4)$ time algorithm that needed to verify over 1500 cases. Recently, Robertson et al. [15] presented an $O(n^2)$ time algorithm that needs to verify just over 600 cases. In what follows we develop an alternate proof that every maximal plane graph admits a face-respecting 2-coloring. Unlike the preceding, our proof embodies a straightforward linear time algorithm to compute the 2-coloring.

We begin by recalling the basic notions of geometric duality and matchings which play a central role in our proofs and in understanding the fundamental properties of face-respecting 2-colorings of maximal plane graphs. Given a plane graph G, the geometric *dual* of G, denoted G^*, is defined as follows: corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G is an edge e^* of G^*; two vertices f^* and g^* of G^* are joined by an edge e^* if and only if the faces f and g of G share the edge e in G.

A subset M of the edges E of a simple graph G is called a *matching* in G if no two elements of M have a vertex in common. A vertex v of G is said to be *matched* by M if it is incident on an element of M. A matching M in G is *maximal* if it cannot be extended to a larger matching of G. A matching M in G is *perfect* if every vertex of G is matched by M.

An edge e forms a *cut edge* in the connected graph G if the removal of e disconnects G. Graph G is said to be 2-*edge connected* if it contains no cut edge. A graph is *k-regular* if all of its vertices have degree k. The following classical result by Petersen (together with its efficient implementation [2]) plays an important role in the development of our algorithms.

Theorem 2.1 ([14], see also [4]). *Every* 2-*edge connected,* 3-*regular graph has a perfect matching.*

A strong connection exists between face-respecting 2-colorings of a maximal plane graph G and perfect matchings on its dual graph G^*. Given any face-respecting 2-coloring of G, let MC denote the set of monochromatic edges of G.

Lemma 2.2. *For any face-respecting* 2-*coloring of a maximal plane graph G the set $\{e^* \mid e \in MC\}$ forms a perfect matching in G^*. Furthermore, for any perfect matching M^* of G^*, there exists a face-respecting* 2-*coloring of G whose monochromatic edges coincide with the set $\{e \mid e^* \in M^*\}$.*

Proof. Let MC^* denote the set $\{e^* \mid e \in MC\}$. First, note that no vertex f^* in G^* is adjacent to two edges in MC^*. Otherwise, the corresponding face f in G would be monochromatic, contradicting our assumptions.

Next, note that every vertex f^* in G^* is adjacent to at least one edge in MC^*. This follows directly from the fact that, for any 2-coloring of G, every face f, being a triangle, must have at least one monochromatic edge. Thus, the set MC^* forms a perfect matching in G^*.

Now let M^* be any perfect matching in G^*. Let M be the edges in G that are dual to the edges in M^* and let G' be the graph G with the edges M removed. Since G is a triangulation, G' is a plane graph each of whose faces is bounded by a 4-cycle. Thus, since every cycle of G' is even, G' is bipartite. Hence G' admits an edge-respecting 2-coloring, which corresponds to a face-respecting 2-coloring of G whose monochromatic edges are precisely those in M. □

Since the dual graph G^* of any maximal plane graph G is both 3-regular and 2-edge-connected, it follows from Petersen's theorem (Theorem 2.1) that G^* admits a perfect matching. Thus, by Lemma 2.2, any maximal plane graph admits a face-respecting 2-coloring. The following algorithm, whose correctness follows directly from the discussion above, constructs such a coloring in linear time, even if G is not maximal.

Algorithm: Face-respecting 2-coloring
Input: An n-vertex plane graph G.
Output: A face-respecting 2-coloring of the vertices of G.
(1) Triangulate G.
(2) Compute the dual G^* of G.
(3) Compute a perfect matching M^* in G^*.
(4) Form graph G' by removing from G edges dual to those in M^*.
(5) Compute an edge-respecting 2-coloring of graph G'.
(6) Output the two color classes.

Theorem 2.2. *The algorithm above constructs a face-respecting* 2*-coloring of an arbitrary* n*-vertex plane graph* G *in* $\mathrm{O}(n)$ *time.*

Proof. Each of the steps can be computed in $\mathrm{O}(n)$ time. Note that the triangulation of step 1 needs to satisfy only topological constraints and so can be achieved without resorting to a more general geometric triangulation algorithm. That step 3 can be performed in $\mathrm{O}(n)$ time was recently shown in [2]. All of the remaining steps are carried out by standard techniques. □

3. Worst-case-minimal vertex and edge covers

Recall that a vertex or an edge of a plane graph G *covers* precisely the faces of G with whose closure it has a non-empty intersection. Note that an edge covers exactly those faces covered by one or more of its endpoints. If a face is covered by *both* endpoints of a given edge it is said to be *strongly* covered by that edge. If a set of vertices or edges covers (respectively strongly covers) all of the faces of G, it is said to cover (respectively strongly cover) G.

In the remainder of this section, we show how to use face-respecting 2-colorings of plane graphs to construct vertex and edge covers and strong covers efficiently.

3.1. Vertex-covers

First, note that each color class of any face-respecting 2-coloring of a plane graph G forms a vertex-cover of G. Since the smaller of these two color classes has size at most $\lfloor n/2 \rfloor$, the following is an immediate consequence of Theorem 2.2.

Theorem 3.1. *Given an* n*-vertex plane graph* G*, a set of at most* $\lfloor n/2 \rfloor$ *vertices that covers* G *can be constructed in* $\mathrm{O}(n)$ *time.*

We know from [3] that there exist n-vertex plane graphs that cannot be covered with fewer than $\lfloor n/2 \rfloor$ vertices.

3.2. Strong edge-covers

Recall from Lemma 2.2 that the monochromatic edges associated with a maximal plane graph G are dual to a perfect matching of the dual graph G^*. Thus, such a set of monochromatic edges forms a strong edge-cover of G of size exactly $n-2$ (half the number of faces of G). Since a strong cover of any maximal plane graph G is easily transformed into a strong cover of any subgraph H of G (by replacing each edge not in H by an edge on the boundary of its associated face), we have the following:

Theorem 3.2. *Given an n-vertex plane graph G, a set of at most $n-2$ edges that strongly covers G can be constructed in $\mathrm{O}(n)$ time.*

Since each edge strongly covers only two faces and every maximal n-vertex plane graph has exactly $2n-4$ faces, this result clearly cannot be improved, in the worst case.

3.3. Edge-covers of maximal plane graphs

Let G be an arbitrary maximal n-vertex plane graph. We describe an algorithm for constructing an edge-cover of G of size $\lfloor n/3 \rfloor$. The first step is to compute a face-respecting 2-coloring of G. Let R and B denote the two color classes. Let $\mathcal{M}(R)$ denote a *maximal* matching on the subgraph of G induced by the vertices in the color class R. The set $\mathcal{M}(R)$ does not itself necessarily cover G. However, if we take all the edges in $\mathcal{M}(R)$ as well as one edge incident on each of the remaining $R - 2|\mathcal{M}(R)|$ unmatched vertices of color R, then the resulting set, denoted $\mathcal{C}(R)$, covers all of the vertices of R and hence covers G. The sets $\mathcal{M}(B)$ and $\mathcal{C}(B)$ are defined similarly. Finally, let $\mathcal{CM} = \mathcal{M}(R) \cup \mathcal{M}(B)$.

Lemma 3.1. *The sets $\mathcal{CM}$, $\mathcal{C}(R)$ and $\mathcal{C}(B)$ form edge-covers of G of total size n.*

Proof. We have already observed that $\mathcal{C}(R)$ (and similarly $\mathcal{C}(B)$) is an edge-cover of G. We now show that $\mathcal{CM}$ is also an edge-cover of G.

Let f be any face of G and suppose, without loss of generality, that f has two vertices in R. Since $\mathcal{M}(R)$ is maximal it follows that at least one of these vertices is incident with an edge e of $\mathcal{M}(R)$. Hence, f is covered by edge e.

Since $|\mathcal{C}(R)| = |R| - |\mathcal{M}(R)|$ and $|\mathcal{C}(B)| = |B| - |\mathcal{M}(B)|$, it follows that:

$$\begin{aligned} \bigl|\mathcal{C}(R)\bigr| + \bigl|\mathcal{C}(B)\bigr| + |\mathcal{CM}| &= |R| - \bigl|\mathcal{M}(R)\bigr| + |B| - \bigl|\mathcal{M}(B)\bigr| + \bigl(\bigl|\mathcal{M}(R)\bigr| + \bigl|\mathcal{M}(B)\bigr|\bigr) \\ &= |R| + |B| \\ &= n. \qquad \square \end{aligned}$$

It is immediate from the above lemma that the smallest of the three sets $\mathcal{C}(R)$, $\mathcal{C}(B)$ and $\mathcal{CM}$ is an edge-cover of size at most $\lfloor n/3 \rfloor$. This establishes the correctness of the following algorithm.

Algorithm: Edge-Cover
Input: An n-vertex maximal plane graph G.
Output: Edge-cover of size at most $\lfloor n/3 \rfloor$.
(1) Compute a face-respecting 2-coloring of graph G.
(2) Compute the sets $\mathcal{C}(R)$, $\mathcal{C}(B)$ and $\mathcal{CM}$.
(3) Output the smaller of the three sets.

Since, all of the steps can be computed in $\mathrm{O}(n)$ time we have the following:

Theorem 3.3. *Given an n-vertex maximal plane graph G, a set of at most $\lfloor n/3 \rfloor$ edges that covers G can be constructed in $\mathrm{O}(n)$ time.*

Bose et al. [3] demonstrate a n-vertex maximal plane graph for which any edge-cover requires at least $\lfloor (4n-4)/13 \rfloor$ edges. It is not known if this bound can be improved. As we will see in the next section, if the requirement of maximality of the plane graph is dropped then $\lfloor n/3 \rfloor$ edges are both necessary and sufficient for edge-covers in the worst case.

3.4. Edge-covers in general plane graphs

The correctness of Algorithm Edge-Cover depends on the assumption that the input graph G is maximal. In this subsection we demonstrate that this assumption can be relaxed so that Theorem 3.3 holds provided only that G contains no quadrilateral faces.

Suppose that H is any plane graph and let G be any triangulation of H. As in the preceding section (where we assumed that $H = G$) we start by computing a face-respecting 2-coloring of G. Let R and B denote the two color classes. Let $\mathcal{M}(R)$ (respectively $\mathcal{M}(B)$) denote a maximal matching on the subgraph of H induced by the vertices in the color class R (respectively B). Let $\mathcal{C}(R)$ denote the set of edges formed by taking all the edges in $\mathcal{M}(R)$ as well as one edge of H incident on each of the remaining $R - 2|\mathcal{M}(R)|$ unmatched vertices of color R. Define $\mathcal{C}(B)$ analogously. Finally, let $\mathcal{CM} = \mathcal{M}(R) \cup \mathcal{M}(B)$. As before, $|\mathcal{C}(R)| = |R| - |\mathcal{M}(R)|$ and $|\mathcal{C}(B)| = |B| - |\mathcal{M}(B)|$, and so the sets $\mathcal{C}(R)$, $\mathcal{C}(B)$ and $\mathcal{CM}$ have total size n.

Although the sets $\mathcal{C}(R)$ and $\mathcal{C}(B)$ still each form edge-covers of H, the set $\mathcal{CM}$ does not necessarily form an edge-cover of H. In fact, if all faces of H are even the set $\mathcal{CM}$ may even be empty (see Fig. 1). The main problem is due to even cycles with no monochromatic edge.

As it turns out, H always admits a triangulation G every face-respecting 2-coloring of which yields a set $\mathcal{CM}$ which forms an edge-cover of all of the non-quadrilateral faces of H. This is clearly the case for faces of H bounded by odd length cycles since every 2-coloring of a cycle with an odd number of vertices must contain at least one monochromatic edge (at least one endpoint of which must be included in any maximal matching of the monochromatic edges of H). Of course, as Fig. 1 illustrates, this is not necessarily true for faces bounded by even cycles. Let f be any face of H bounded by a cycle $C = v_1, v_2, \ldots, v_k$ with $k \geqslant 6$ vertices. Add the edges $b_1 = [v_1, v_3]$, $b_2 = [v_3, v_5]$ and $b_3 = [v_5, v_1]$ to the interior of f such that the three edges subdivide f into three triangles and a face of size $k - 3$. We refer to this operation as *bridging* the face f (see Fig. 2).

Let H' be any plane graph formed from H by bridging all faces with six or more vertices. Then any triangulation G of H' (which, of course, is also a triangulation of H) has the property that any face-

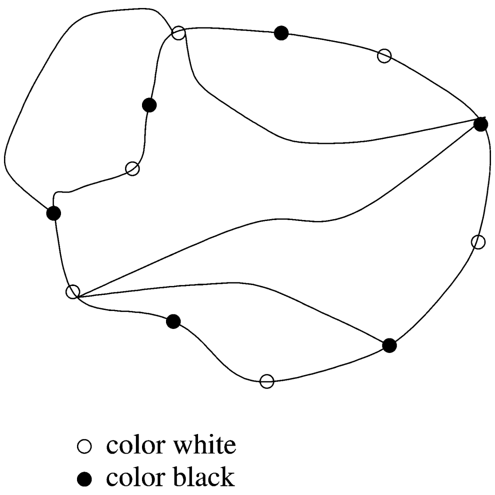

Fig. 1. Face-respecting 2-coloring where $\mathcal{CM}$ is empty.

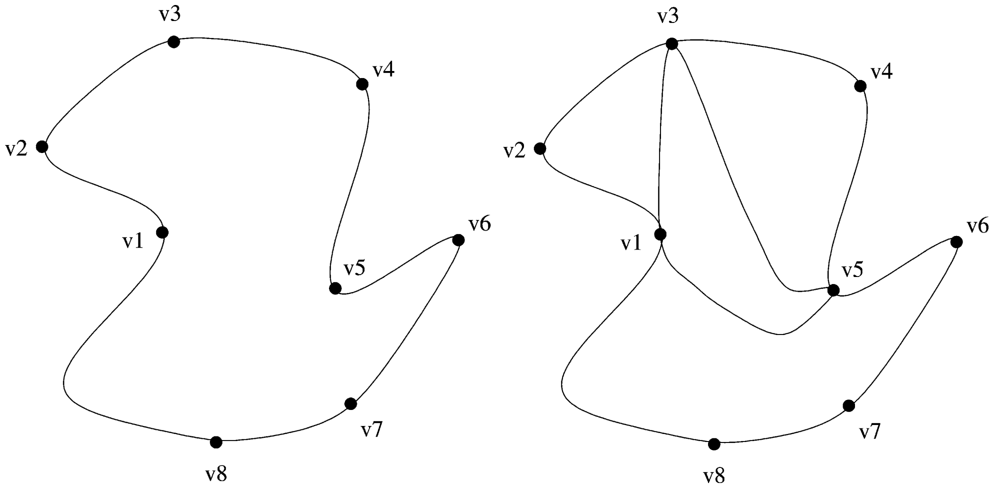

Fig. 2. Bridging a cycle.

respecting 2-coloring of H produces at least one monochromatic edge on every non-quadrilateral face of H (the only 2-colorings that might avoid this in a given face are precluded by the bridge edges). Thus, if H contains no quadrilateral faces then any triangulation G of H' has the property that every face-respecting 2-coloring of G yields a set $\mathcal{CM}$ which forms an edge-cover of H. Moreover, if H contains q quadrilateral faces, the set $\mathcal{CM}$ can be augmented by one edge incident with each quadrilateral face (at most q in total) to form an edge-cover of H.

Our algorithm to compute an edge-cover in an arbitrary plane graph H is essentially the same as Algorithm Edge-Cover. The only difference is that each even face (of size six or greater) of the graph must be bridged before computing the face-respecting 2-coloring. This guarantees that $\mathcal{CM}$ (together with at most one edge incident with each quadrilateral face) is an edge-cover of H. Since bridging all of the even faces of a plane graph can be achieved in linear time, we have the following:

Theorem 3.4. *Given an n-vertex plane graph H with q quadrilateral faces, a set of at most $\lfloor n/3 \rfloor + q$ edges that cover H can be constructed in* $\mathrm{O}(n)$ *time.*

An example of an n-vertex plane graph (with no quadrilateral faces) for which any edge-cover requires $\lfloor n/3 \rfloor$ edges is given in [3].

It is straightforward to interpret the covering results of this section in terms of utility location on (the vertices or edges of) general planar subdivisions. In the next section, we set out in more detail the application of our covering results to problems concerning the placement of guards (or light sources) on polyhedral surfaces.

4. Applications to guarding of polyhedral surfaces

Let P be an arbitrary polyhedron of genus zero. The boundary of P, denoted $\partial(P)$, defines a *polyhedral surface*. We denote by $\sigma_1(P)$ the 1-skeleton of P, that is the planar graph whose vertices (respectively edges) are the 0-faces (respectively 1-faces) of P. We denote by $\sigma_2(P)$ the 2-skeleton associated with P; that is the plane graph whose vertices (respectively edges and faces) are the 0-faces (respectively 1-faces and 2-faces) of P.

A point a on the polyhedral surface $\partial(P)$ is said to be *visible* from the point x on $\partial(P)$ if the line segment $[ax]$ does not intersect any point in the interior of P. The set of all such points a is called the *visible region* associated with x. More generally, the visible region associated with the point set X on $\partial(P)$ is just the union of the visible regions associated with points x in X.

We restrict our attention to sets of guards comprised of points or straight line segments of $\partial(P)$. A *vertex-guard* is a guard that coincides with a single vertex (0-face) of $\partial(P)$. An *edge-guard* is a guard that coincides with an edge (1-face) of $\partial(P)$. A *face-diagonal-guard* is a guard that coincides with a diagonal of a 2-face of P. Note that the visible region associated with either a vertex-guard, an edge-guard or a face-diagonal-guard contains the union of all faces whose closure has a non-empty intersection with that guard and, when the underlying polyhedron P is convex, that is all that it contains.

A set of guards forms a *guard-set* for any subset of its associated visible region. It is natural to try and identify minimum-size guard-sets for $\partial(P)$ (or for specified subsets of $\partial(P)$). In light of the observations above, it is convenient to focus on the underlying combinatorial problem of covering the associated plane graph $\sigma_2(P)$. Note that since not every plane graph is the 2-skeleton of a convex polyhedron, the problem of finding minimum-size guard-sets for general polyhedral surfaces is not necessarily equivalent to the combinatorial problem of finding minimum-size covers for plane graphs. However, upper bounds on the size of covers for plane graphs with n vertices provide upper bounds on the size of guard-sets for polyhedral surfaces with n vertices.

Observation 4.1. *A cover by vertices* (*respectively edges or face-diagonals*) *of* (*a specified subset of*) *the* 2*-skeleton of a polyhedron P provides a vertex-guard-set* (*respectively edge-guard-set or face-diagonal-guard-set*) *for* (*the corresponding subset of*) *the polyhedral surface* $\partial(P)$.

Note that a *polyhedral surface* is a generalization of a *polyhedral terrain*. Therefore, all of the results for polyhedral surfaces also hold for polyhedral terrains. From the results of the preceding section and Observation 4.1 we have the following corollaries.

Corollary 4.1. *Given a polyhedral surface $\partial(P)$ with n vertices, a vertex-guard-set of $\partial(P)$, of size at most $\lfloor n/2 \rfloor$, can be constructed in $\mathrm{O}(n)$ time.*

Corollary 4.2. *Given a polyhedral surface $\partial(P)$ with n vertices and q quadrilateral faces, an edge-guard-set of $\partial(P)$, of size at most $\lfloor n/3 \rfloor + q$, can be constructed in $\mathrm{O}(n)$ time.*

Corollary 4.3. *Given a polyhedral surface $\partial(P)$ with n vertices and q quadrilateral faces, a face-diagonal-guard-set of $\partial(P)$, of size at most $\lfloor n/3 \rfloor$, can be constructed in $\mathrm{O}(n)$ time. Furthermore, the number of proper face-diagonal-guards (face-diagonals that are not edges of $\partial(P)$) in such a set need not exceed q.*

Corollary 4.4. *Given a polyhedral surface $\partial(P)$ with n vertices, a strong edge-guard-set of $\partial(P)$, of size at most $n - 2$, can be constructed in $\mathrm{O}(n)$ time.*

5. Summary and open problems

We have presented simple linear-time algorithms that determine, for an arbitrary n-vertex plane graph G, a vertex-cover for G of size at most $\lfloor n/2 \rfloor$ or an edge-cover for G of size at most $\lfloor n/3 \rfloor$ plus the number of quadrilateral faces of G. We also outlined some applications of these results.

Two interesting questions deserve further consideration. First, the algorithm used to compute the edge-covers of an arbitrary plane graph G is sensitive to the number of quadrilateral faces of G. It would be interesting to remove this dependence.

It was shown in [8] (and again here) that $\lfloor n/3 \rfloor$ edges are always sufficient to cover an n vertex plane graph. The best corresponding lower bound is provided by a family of maximal plane graphs whose minimum edge-covers have size $\lfloor 4n - 4/13 \rfloor$ [3]. While it is quite simple to construct a 2-connected plane graph that requires $\lfloor n/3 \rfloor$ edge guards (see [3]), we know of no such graph that is the 2-skeleton of a polyhedral surface. It would be sufficient to construct a plane triangulation requiring $\lfloor n/3 \rfloor$ edge guards.

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